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ESTIMATING LOCATION AND SCALE PARAMETERS OF A SYMMETRIC DISTRIBUTION BY SYSTEMATIC STATISTICS

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SUMMARY

It is shown how the usual best linear unbiased estimate of location and scale parameters of a symmetric distribution simplifies into linear functions of systematic statistics viz, quasi-midranges or quasi-ranges of the sample. The scale parameter of a Cauchy distribution has been obtained using quasi-ranges for certain small samples.

Keywords : Location and scale parameters; Cauchy distribution; Quasi-midranges; Quasi-ranges; order statistics.

Introduction

Lloyd [2] has introduced the method of obtaining the Best Linear Unbiased Estimate (BLUE) of location and scale parameters by order statistics. If $X_{(1)} \leq X_{(2)} \leqslant \ldots < X_{(n)}$, is the order statistics of a sample of size n, then

$$M_i = (X_{(n-i+1)} + X_{(i)})/2 \; , \; i = 1, 2, \ldots , [(n+1)/2], \quad (1.1)$$

is known as the ith quasi-midrange and

$$R_i = X_{(n-i+1)} - X_{(i)}; \; i = 1, 2, \ldots , [n/2], \quad (1.2)$$

the ith quasi-range of the sample, where [.] is the usual greatest integer function. In Section 2, it is shown how the BLUE of the Location and Scale parameters of a symmetric distribution reduces to that based on quasi-midranges or quasi-ranges of the sample. In sections 3 and 4, the estimate of location and scale parameters based on quasi-midranges and quasi-ranges respectively are derived with the exact expression for their variance.

For almost all distributions belonging to the location-scale family, the BLUE of the location and scale parameters have been evaluated for small samples. However the estimate of the scale parameter of a Cauchy distribution is not yet seen published like others. Hence in Section 5, the estimate of the scale parameter of a Cauchy distribution based on quasi-ranges and its variance are obtained for $n = 6\ (1)\ 16\ (2)\ 20$.

2. Motivation for Estimating the Parameters by Quasi–Midranges and Quasi-Ranges

Let $X_{(1)} < X_{(2)} \leqslant \ldots < X_{(n)}$ be the order statistics of a sample of size n, drawn from an absolutely continuous distribution symmetric about the location parameter μ and with a scale parameter σ. Let

$$Y_{(r)} = (X_{(r)} - \mu)\sigma,\ r = 1, 2, \ldots, n \tag{2.1}$$

Then $Y_{(r)}$, $r = 1,2, \ldots, n$ becomes the order statistics of a distribution which is free from μ and σ. As the basic distribution is symmetric about μ, the distribution of $Y' = (Y_{(1)}, Y_{(2)}, \ldots, Y_{(n)})$ and $(-Y_{(n)}, -Y_{(n-1)}, \ldots, -Y_{(1)})$ are identical. Also

$$\begin{bmatrix} -Y_{(n)} \\ -Y_{(n-1)} \\ \cdot \\ \cdot \\ \cdot \\ -Y_{(1)} \end{bmatrix} = -J \begin{bmatrix} Y_{(1)} \\ Y_{(2)} \\ \cdot \\ \cdot \\ \cdot \\ Y_{(n)} \end{bmatrix} \quad \text{where } J = \begin{bmatrix} 0\,0 \ldots 0\,1 \\ 0\,0 \ldots 1\,0 \\ \cdot \\ \cdot \\ \cdot \\ 1\,0 \ldots 0\,0 \end{bmatrix}$$

Let the expectation of the vector Y and its covariance matrix $D(Y)$ be given by

$$E(Y) = a \tag{2.2}$$

and $\mathcal{D}(Y) = A$ (2.3)

Clearly the elements of a and A are known. Since we have $E(-JY) = -Ja$, we have

$$a = -Ja \quad (2.4$$

Also $D(-JY) = JAJ$ and hence

$$A = JAJ \quad (2.5)$$

Left multiplication of A by J changes the ith row into $(n-i+1)$th row and right multiplication of A by J changes the jth column into $(n-j+1)$th column. Then from (2.5) it is seen that if the jth column vector of A is $(c_1, c_2, \ldots, c_n)'$, then its $(n-j+1)$th column vector is $(c_n, c_{n-1}, \ldots, c_1)'$. Similarly if the ith row of A is $(r_1, r_2, \ldots, r_n)$ then its $(n-i+1)$th row is $(r_n, r_{n-1}, \ldots, r_1)$. Also from the identity $J = J^{-1} = J'$ we get

$$A^{-1} = JA^{-1}J \quad (2.6)$$

Thus we see that the property described for A is true with A^{-1} also. From (2.1) to (2 3) we write (2.1)

$$E(X) = 1\mu + a\sigma, \quad \text{where}$$

$$X' = (X_{(1)}, X_{(2)}, \ldots, X_{(n)}),$$

$1' = (1, 1, \ldots, 1)$ and $D(X) = A\sigma^2$. Then applying Gauss Markov-theorem the BLUE of μ and σ are given by

$$\mu^* = (1'A^{-1}1)^{-1} \cdot 1'A^{-1}X, \quad (2.7)$$

$$\sigma^* = (a'A^{-1}a)^{-1} \cdot a'A^{-1}X, \quad (2.8)$$

$$\text{var}(\mu^*) = (1'A^{-1}1)^{-1}\sigma^2,$$

$$\text{var}(\sigma^*) = (a'A^{-1}a)^{-1}\sigma^2,$$

$$\text{Cov}(\mu^*,) = 0$$

from (2.6) and the Lloyd's BLUE (2.7) of μ it is seen that apart from the constant multiple $(1'A^{-1}1)^{-1}$, the coefficient of $X_{(i)}$ in the estimate of μ is obtained just by adding the elements of ith column of A^{-1} and obviously this sum is same as the sum of elements of $(n-i+1)$ th column. This property of μ^* proves that Lloyd's BLUE of μ reduces to a linear function of sample quasi-mid-ranges.

Obviously the form of the vector a is $a' = (-a_1, \ldots, -a_{[n/2]}, a_{[n/2]}, \ldots, a_1)$ for n even and $a' = (-a_1, \ldots, -a_{[n/2]}, 0, a_{[n/2]}, \ldots, a_1)$ for n odd where $[\cdot]$ is the usual greatest integer function Using this property of a and equations (2.6) and (2.8), we see that the coefficients of $X_{(j)}$ and $X_{(n-j+1)}$ in the estimate σ^* are equal in magnitude but opposite in sign. This property of σ^* proves that, Lloyd's BLUE of σ reduces to a linear function of sample quasi-ranges.

Thus, for symmetric populations Lloyd's BLUE reduces to a linear function of quasi-mid-ranges for estimating μ and to a linear function of quasi-ranges for estimating σ.

3. Estimating μ by Quasi Midranges

Let $M_1, M_2, \ldots, M_{[(n+1)/2]}$ be the quasi-mid-ranges of a sample of size n. From (1.1) and using the distributional symmetry we have,

$$E(M_i) = \mu$$

$$\text{Cov}(M_i, M_j) = 2^{-1}(a_{i,j} + a_{i,n-j+1})\sigma^2 \quad \text{where}$$

$$a_{i,j} = \text{Cov}(Y_i, Y_j)$$

Thus if $M = (M_1, M_2, \ldots, M_{[(n+1)/2]}$

then $E(M) = 1\mu$, where $1' = (1, 1, \ldots, 1)$

$$D(M) = V\sigma^2$$

where $V = ((v_{i,j}))$ and $v_{i,j} = \text{Cov}(M_i, M_j)$

Then by applying the Gauss-Markov theorem we get the BLUE of μ based on quasi-mid-ranges as

$$\mu^* = (1'V^{-1}1)^{-1}1'V^{-1}M \tag{3.1}$$

with $\text{Var}(\mu^*) = (1'V^{-1}1)^{-1}\sigma^2$ (3.2)

The advantage of using quasi-midranges to estimate μ is that the order of covariance matrix involved in (3.2) is only $[(n+1)/2]$ whereas in Lloyd's estimate (2.7), the covariance matrix involved is A with order n.

4. Estimating σ by Quasi-Ranges

Let $R_1, R_2, \ldots, R_{[n/2]}$ be the quasi-ranges of a sample of size n. From (1.2) and using the distributional symmetric properties we get

$$E(R_i) = 2a_i\,\sigma, \text{ where } a_i = E(Y_{(n-i+1)})$$

$$\text{Cov}(R_i\, R_j) = 2\,(a_{i,j} - a_{i,\,n-j+1})\,\sigma^2;$$

where $\text{Cov}(Y_i, Y_j) = a_{i,j}$.

Let $R' = (R_1, R_2, \ldots, R_{[n/2]})$ then

$$E(R) = b\,\sigma,$$

where $b' = (2a_1, 2a_2, \ldots, 2a_{[n/2]})$

$$D(R) = w\,\sigma^2$$

where $W = ((W_{i,j}))$ and $W_{i,j} = \text{Cov}(R_i, R_j)$

Then from Gauss-Markov theorem we have

$$\sigma^* = (b'\ W^{-1}b)^{-1}\, b'\ W^{-1}R, \tag{4.1}$$

and

$$\text{Var}(\sigma^*) = (b'\ W^{-1}b)^{-1}\,\sigma^2 \tag{4.2}$$

As in the case of estimating μ here again the order of the covariance matrix W is only $[n/2]$ whereas in Lloyd's estimate of σ, the covariance martix involved is A with order n.

5. Estimating the Scale Parameter of Cauchy Distribution

Let $R_1, R_2, \ldots, R_{[n/2]}$ be the quasi-ranges of a sample of size n drawn from a Cauchy distribution.

$$f(x, \mu, \sigma) = (\pi\,\sigma)^{-1}\,(1 + ((x-\mu)/\sigma)^2)^{-1},$$

TABLE 1—BLUE OF THE SCALE PARAMETER σ OF CAUCHY DISTRIBUTION BY QUASI-RANGES

n	Coefficients of BLUE $\sigma^* = \sum_{i=3}^{[n/2]} c_i R_i$								$Var(\sigma^*)/\sigma^2$
	C_3	C_4	C_5	C_6	C_7	C_8	C_9	C_{10}	
6	1.3820								1.7552
7	0.7898								1.0032
8	0.4135	0.5780							0.6848
9	0.2338	0.5611							0.5183
10	0.1417	0.4291	0.3007						0.4158
11	0.0911	0.3117	0.3716						0.3467
12	0.0613	0.2258	0.3460	0.1801					0.2969
13	0.0428	0.1657	0.2930	0.2535					0.2595
14	0.0308	0.1232	0.2397	0.2654	0.1204				0.2303
15	0.0228	0.0933	0.1940	0.2484	0.1815				0.2070
16	0.0172	0.0719	0.1560	0.2213	0.2055	0.0828			0.1879
18	0.0104	0.0444	0.1028	0.1645	0.1940	0.1597	0.0633		0.1586
20	0.0067	0.0289	0.0693	0.1190	0.1593	0.1672	0.1273	0.0489	0.1371

$$\sigma > 0, \; -\infty < X < \infty$$

where μ and σ are the location and scale parameters respectively. Barnett [1] evaluated all existing means variances, convariances of order statistics and Lloyd's BLUE of μ for $n = 5$ (1) 16 (2) 20. As the variances of the first two and last two order statistics of a Cauchy distribution do not exists, the Barnett's estimate of μ utilizes only the order statistics $X_{(4)}, X_{(3)}, \ldots, X_{(n-2)}$.

The Cauchy distribution does not possesses the first two moments and as a result it is not possible to estimate μ and σ by the method of moments. Moreover the exact variance of the maximum likelihood estimates are also not known for small samples drawn from this distribution. For these reasons the method of estimating μ and σ of a Cauchy distribution by order statistics attracts special importance in small sample case.

As the BLUE of μ based on quasi-ranges can be obtained from Barnett's tabulated estimate, it needs no further evaluation. By using (4.1) and (4.2), the BLUE of σ based on quasi-ranges $R_3, R_4, \ldots, R_{[n/]}$ and its variance are obtained for $n = 6$ (1) 16 (2) 20 and presented in Table 1. In the table the estimate of σ as given in (4.1) is taken as

$\sigma^* = \sum_{i=3}^{[n/2]} C_i R_i$ and the c_i values are given along the columns.

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